Biometrika (1981), **68**, 1, *pp.* 165–76
Printed in Great Britain

Fractional differencing

BY J. R. M. HOSKING
Institute of Hydrology, Wallingford, Oxfordshire

SUMMARY

The family of autoregressive integrated moving-average processes, widely used in time series analysis, is generalized by permitting the degree of differencing to take fractional values. The fractional differencing operator is defined as an infinite binomial series expansion in powers of the backward-shift operator. Fractionally differenced processes exhibit long-term persistence and antipersistence; the dependence between observations a long time span apart decays much more slowly with time span than is the case with the more commonly studied time series models. Long-term persistent processes have applications in economics and hydrology; compared to existing models of long-term persistence, the family of models introduced here offers much greater flexibility in the simultaneous modelling of the short-term and long-term behaviour of a time series..

Some key words: Autoregressive integrated moving-average process; Fractional differencing; Long-term persistence; Time series.

1. INTRODUCTION

Most of the recent work in time series analysis has been concerned with series having the property that observations separated by a long time span are independent or nearly so. Yet in many empirical time series, particularly those arising in economics and hydrology, the dependence between distant observations, though small, is by no means negligible. Such series appear to exhibit cycles and changes of level of all orders of magnitude, and their spectral densities increase indefinitely as the frequency decreases to zero (Adelman, 1965; Granger, 1966). Indeed, such 'long-term persistence' is perhaps best characterized by the occurrence of a spectral density behaving like ω^d, $d < 0$, as $\omega \to 0$ (Cox, 1977).

The practical use of long-term persistent time series models has been described by Lawrance & Kottegoda (1977). As yet, however, there is still a need for a family of models which have all the desirable properties of:

(1) explicitly modelling long-term persistence;
(2) being flexible enough to explain both the short-term and long-term correlation structure of a series;
(3) enabling synthetic series to be easily generated from the model.

The aim of the present paper is to introduce a family of models which does meet these requirements, by generalizing the well-known ARIMA (p, d, q) models of Box & Jenkins (1976). The generalization consists of permitting the degree of differencing d to take any real value rather than being restricted to integral values; it turns out that for suitable values of d, specifically $0 < d \leqslant \frac{1}{2}$, these 'fractionally differenced' processes are capable of modelling long-term persistence.

An outline of the paper is as follows. In § 2 we derive heuristically the fundamental fractionally differenced process, the ARIMA $(0, d, 0)$ process, also known as fractionally differenced white noise. In §3 we give some of the basic properties of the ARIMA $(0, d, 0)$ process and describe in more detail the behaviour of the process for values of d in the range $-\frac{1}{2} \leqslant d \leqslant \frac{1}{2}$. In §4 the general ARIMA (p, d, q) family is defined and in §5 the properties of some of the simpler processes in the family are described. Section 6 contains an outline of a procedure for identification of fractionally differenced models and comments on some other possible generalizations of ARIMA (p, d, q) processes.

2. DERIVATION OF FRACTIONALLY DIFFERENCED WHITE NOISE

Brownian motion is a continuous time stochastic process $B(t)$ with independent Gaussian increments and spectral density ω^{-2}. Its derivative is the continuous-time white noise process, which has constant spectral density. Fractional Brownian motion, $B_H(t)$, defined by Mandelbrot (1965) and Mandelbrot & van Ness (1968), is a generalization of these processes. The basic properties of fractional Brownian motion are:

(i) fractional Brownian motion with parameter H, usually $0 < H < 1$, is the $(\frac{1}{2}-H)$th fractional derivative of Brownian motion, the derivative being defined in the Weyl or Riemann–Liouville senses (Mandelbrot & van Ness, 1968);
(ii) the spectral density of fractional Brownian motion is proportional to ω^{-2H-1};
(iii) the covariance function of fractional Brownian motion is proportional to $|k|^{2H-2}$.

The continuous-time fractional noise process is then defined as $B'_H(t)$, the derivative of fractional Brownian motion; it may also be thought of as the $(\frac{1}{2}-H)$th fractional derivative of continuous-time white noise, to which it reduces when $H = \frac{1}{2}$. Technically, the derivative exists only in the sense of a random Schwartz distribution, but the mathematical details of continuous time processes do not concern us here.

We seek a discrete time analogue of continuous-time fractional noise. One possibility is discrete-time fractional Gaussian noise, proposed by Mandelbrot & Wallis (1969), which is defined to be a process whose correlation function is the same as that of the process of unit increments $\Delta B_H(t) = B_H(t) - B_H(t-1)$ of fractional Brownian motion. Here, however, we investigate a different approach; we look for a discrete time version of (i) rather than (iii) of the properties of fractional Brownian motion given above.

The discrete time analogue of Brownian motion is the random walk, or ARIMA $(0, 1, 0)$ process, $\{x_t\}$, defined by

$$\nabla x_t = (1-B)\, x_t = a_t,$$

where B is the backward-shift operator defined by $Bx_t = x_{t-1}$ and the a_t are independent identically distributed random variables. The first difference of $\{x_t\}$ is the discrete-time white noise process $\{a_t\}$. By analogy with the above definition of continuous-time white noise we define fractionally differenced white noise with parameter H to be the $(\frac{1}{2}-H)$th fractional difference of discrete-time white noise. The fractional difference operator ∇^d is defined in the natural way, by a binomial series:

$$\nabla^d = (1-B)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-B)^k = 1 - dB - \tfrac{1}{2}d(1-d)\, B^2 - \tfrac{1}{6}d(1-d)(2-d)\, B^3 - \dots . \tag{2·1}$$

We write $d = H - \frac{1}{2}$, so that continuous-time fractional noise with parameter H has as its discrete time analogue the process $x_t = \nabla^{-d} a_t$, or $\nabla^d x_t = a_t$, where $\{a_t\}$ is a white noise process. We call $\{x_t\}$ an ARIMA $(0, d, 0)$ process, a natural extension of the terminology of Box & Jenkins (1976) to nonintegral d. Apart from a passing reference by Granger (1978), fractional differencing does not appear to have been previously mentioned in connexion with time series analysis.

3. The ARIMA $(0, d, 0)$ process

We formally define an ARIMA $(0, d, 0)$ process to be a discrete-time stochastic process $\{x_t\}$ which may be represented as

$$\nabla^d x_t = a_t,$$

where the operator ∇^d is as defined in (2·1) and the white noise process $\{a_t\}$ consists of independent identically distributed random variables with mean zero and variance σ_a^2. The following theorem gives some of the basic properties of the process, assuming for convenience that $\sigma_a^2 = 1$.

THEOREM 1. *Let $\{x_t\}$ be an* ARIMA $(0, d, 0)$ *process.*

(a) *When $d < \frac{1}{2}$, $\{x_t\}$ is a stationary process and has the infinite moving-average representation*

$$x_t = \psi(B)\, a_t = \sum_{k=0}^{\infty} \psi_k\, a_{t-k},$$

where

$$\psi_k = \frac{d(1+d)\ldots(k-1+d)}{k!} = \frac{(k+d-1)!}{k!\,(d-1)!}. \tag{3·1}$$

As $k \to \infty$, $\psi_k \sim k^{d-1}/(d-1)!$.

(b) *When $d > -\frac{1}{2}$, $\{x_t\}$ is invertible and has the infinite autoregressive representation*

$$\pi(B)\, x_t = \sum_{k=0}^{\infty} \pi_k\, x_{t-k} = a_t,$$

where

$$\pi_k = \frac{-d(1-d)\ldots(k-1-d)}{k!} = \frac{(k-d-1)!}{k!(-d-1)!}.$$

As $k \to \infty$, $\pi_k \sim k^{-d-1}/(-d-1)!$. In parts (c)–(f), *we assume that $-\frac{1}{2} < d < \frac{1}{2}$.*

(c) *The spectral density of $\{x_t\}$ is $s(\omega) = (2 \sin \frac{1}{2}\omega)^{-2d}$ for $0 < \omega \leqslant \pi$ and $s(\omega) \sim \omega^{-2d}$ as $\omega \to 0$.*

(d) *The covariance function of $\{x_t\}$ is*

$$\gamma_k = E(x_t\, x_{t-k}) = \frac{(-1)^k\,(-2d)!}{(k-d)!\,(-k-d)!} \tag{3·2}$$

and the correlation function of $\{x_t\}$ *is*

$$\rho_k = \gamma_k/\gamma_0 = \frac{(-d)!\,(k+d-1)!}{(d-1)!\,(k-d)!} \quad (k = 0, \pm 1, \ldots),$$

$$\rho_k = \frac{d(1+d)\ldots(k-1+d)}{(1-d)(2-d)\ldots(k-d)} \quad (k = 1, 2, \ldots).$$

In particular $\gamma_0 = (-2d)!/\{(-d)!\}^2$ *and* $\rho_1 = d/(1-d)$. *As* $k \to \infty$,

$$\rho_k \sim \frac{(-d)!}{(d-1)!} k^{2d-1}. \tag{3·3}$$

(e) *The inverse autocorrelations of* $\{x_t\}$ *are*

$$\rho_{\mathrm{inv},k} = \frac{d!\,(k-d-1)!}{(-d-1)!\,(k+d)!} \sim \frac{d!}{(-d-1)!} k^{-1-2d}$$

as $k \to \infty$.

(f) *The partial correlations of* $\{x_t\}$ *are*

$$\phi_{kk} = d/(k-d) \quad (k = 1, 2, \ldots). \tag{3·4}$$

Proof. (a) Writing $x_t = \psi(B)\,a_t$, we have $\psi(z) = (1-z)^{-d}$. When $d < \frac{1}{2}$ the power series expansion of $\psi(z)$ converges for $|z| \leqslant 1$ and so $\{x_t\}$ is stationary. Binomial expansion of $(1-z)^{-d}$ gives (3·1). As $k \to \infty$, $(k+d-1)!/k! \sim k^{d-1}$ by Stirling's formula.

(b) The proof is similar to (a) but with d replaced by $-d$.

(c) Since $\sigma_a^2 = 1$, we have $s(\omega) = \psi(e^{i\omega})\,\psi(e^{-i\omega})$. The result follows on substitution of $\psi(z) = (1-z)^{-d}$.

(d) Since $\sigma_a^2 = 1$, we have

$$\gamma_k = \pi^{-1} \int_0^\pi \cos(k\omega)\, s(\omega)\, d\omega.$$

Result (3·2) now follows from a standard result for

$$\int_0^\pi (\sin x)^{v-1} \cos(ax)\, dx$$

(Gradshteyn & Ryzhik, 1965, p. 372). The other results follow immediately.

(e) The inverse correlations of the ARIMA $(0, d, 0)$ process $\nabla^d x_t = a_t$ are the same as the correlations of the ARIMA $(0, -d, 0)$ process $y_t = \nabla^d a_t$ (Chatfield, 1979, p. 366). Thus the result follows from part (d) of the theorem.

(f) The relationship between correlations and partial correlations is given by the Levinson–Durbin algorithm (Box & Jenkins, 1976, p. 84). Values for the partial correlations $\phi_{kk} = d/(k-d)$ and the partial linear regression coefficients

$$\phi_{kj} = -\binom{k}{j} \frac{(j-d-1)!\,(k-d-j)!}{(-d-1)!\,(k-d)!}$$

may be proved by induction on k using the algorithm.

From the theorem we see that when $-\frac{1}{2} < d < \frac{1}{2}$ the process $\{x_t\}$ is both stationary and invertible. Both ψ_k and π_k decay hyperbolically, rather than showing the exponential decay characteristic of an ARIMA $(p, 0, q)$ process. The behaviour of the spectrum at low frequencies indicates that for $d > 0$ $\{x_t\}$ is a long-term persistent process. Long-term persistence may also be characterized by the hyperbolic decay (3·3) of the correlation function; (3·3) also implies that $\{x_t\}$ is asymptotically self-similar, as defined by Mandelbrot (1965). Unlike the case of an ARIMA $(p, 0, q)$ process, the partial correlations and inverse correlations of $\{x_t\}$ decay hyperbolically and at different rates; indeed the partial correlations of $\{x_t\}$ decay as k^{-1}, which is independent of d. This seems surprising at first sight, but the distinction between different values of d is in fact made by the behaviour of the partial linear regression coefficients ϕ_{kj} for $1 \leqslant j \leqslant k$: we have, as $j, k \to \infty$ with $j/k \to 0$,

$$\phi_{kj} \sim -j^{-d-1}/(-d-1)!.$$

McLeod & Hipel (1978) define a stationary process as having a long or short memory according to whether its correlations have an infinite or a finite sum. Theorem 1 implies that for $0 < d < \frac{1}{2}$ the ARIMA $(0, d, 0)$ process is a long-memory stationary process.

An ARIMA $(0, d, 0)$ process, where d is any real number, may be summed or differenced a finite integral number of times until d lies in the interval $[-\frac{1}{2}, \frac{1}{2}]$ and will then be stationary and invertible, except if $d = \pm\frac{1}{2}$, when the final process will be either stationary or invertible but not both. This range is the most useful set of values of d, so that we consider these processes in more detail.

When $d = \frac{1}{2}$, the spectral density of the process is

$$s(\omega) = 1/\{2\sin(\tfrac{1}{2}\omega)\} \sim \omega^{-1}$$

as $\omega \to 0$. Thus the ARIMA $(0, \frac{1}{2}, 0)$ process is a discrete-time '$1/f$ noise' (Mandelbrot, 1967). The formal infinite moving-average representation of the process, $x_t = (1-B)^{-\frac{1}{2}}a_t = \Sigma_k \psi_k a_{t-k}$, has $\psi_k \sim (\pi k)^{-\frac{1}{2}}$ as $k \to \infty$. Thus $\Sigma\psi_k^2$ 'just fails to converge', so that $\{x_t\}$ is 'just nonstationary'. However, $\{x_t\}$ is invertible: we have

$$(1 - \tfrac{1}{2}B - \tfrac{1}{8}B^2 - \tfrac{1}{16}B^3 - \tfrac{5}{128}B^4 - \ldots)\,x_t = a_t,$$

the weights π_k behaving like $-\frac{1}{2}\pi^{-\frac{1}{2}}k^{-3/2}$ as $k \to \infty$.

When $0 < d < \frac{1}{2}$, the ARIMA $(0, d, 0)$ process is a stationary process with long memory, and as such may be expected to be useful in modelling long-term persistence. The correlations and partial correlations of $\{x_t\}$ are all positive and decay monotonically and hyperbolically to zero as the lag increases. The spectral density of $\{x_t\}$ is concentrated at low frequencies: $s(\omega)$ is a decreasing function of ω and $s(\omega) \to \infty$ as $\omega \to 0$, but $s(\omega)$ is integrable. At low frequencies, $\{x_t\}$ has the spectrum of 'affine noise' (Mandelbrot, 1977, p. 277); the spectrum as a whole has a shape 'typical of an economic variable' (Granger, 1966).

When $d = 0$, the ARIMA $(0, 0, 0)$ process is white noise, with zero correlations and constant spectral density.

When $-\frac{1}{2} < d < 0$, the ARIMA $(0, d, 0)$ process has a short memory, and is 'antipersistent' in the terminology of Mandelbrot (1977, p. 232). The correlations and partial correlations of the process are all negative, except $\rho_0 = 1$, and decay monotonically and hyperbolically to zero. The spectral density is dominated by high-frequency components; $s(\omega)$ is an increasing function of ω, and vanishes at $\omega = 0$ but has gradient $+\infty$ there.

The ARIMA $(0, -\frac{1}{2}, 0)$ process is stationary but not invertible, so that forecasts of the process cannot be expressed as a convergent sum of past values of the process. The ψ_k weights of the infinite moving-average representation of the process are the same as the π_k weights of an ARIMA $(0, \frac{1}{2}, 0)$ process and thus decay as $k^{-3/2}$ for large k. The spectral density of the process is $s(\omega) = 2\sin(\frac{1}{2}\omega)$; it tends to zero as $\omega \to 0$, but, unlike the case $-\frac{1}{2} < d < 0$, has finite gradient $+1$ at $\omega = 0$. The correlation function of the process is $\rho_k = -1/(4k^2-1)$; the process variance is $\gamma_0 = 4/\pi$. The partial correlations of the process are given by $\phi_{kk} = -1/(2k+1)$.

4. THE ARIMA (p, d, q) PROCESS

The ARIMA $(0, d, 0)$ process, being a form of fractional noise, is comparable to the fractional Gaussian noise process of Mandelbrot & Wallis (1969), and may be expected to have similar applications in time series modelling (Wallis & Matalas, 1970). Yet in practical problems of fitting time series models to hydrological data, the modelling ability of fractional Gaussian noise has sometimes been claimed to be inferior to that of other processes; see, for example, Hipel & McLeod (1978). This seems to be because fractional Gaussian noise, having only three variable parameters, mean, variance and H, is not flexible enough to model the wide range of low-lag correlation structures encountered in practice; and the methods commonly used to compare modelling procedures, i.e. accuracy of short-term forecasts, values of the Akaike information criterion, resemblance of the correlation function of the fitted model to that of the observed series, give little emphasis to the purpose for which fractional Gaussian noise was intended, namely modelling the long-term behaviour of an observed series. What is required, then, is an extension of the fractional-noise model to encompass a wider range of short-term behaviour while retaining the eventual hyperbolic decay of the correlation function. There does not appear to be any suitable simple modification of fractional Gaussian noise, although some *ad hoc* methods have been suggested, such as the filtered moving-average process of Matalas & Wallis (1971). But there is a very natural extension of the ARIMA $(0, d, 0)$ process which has the required properties; we can combine fractional differencing with the established family of Box–Jenkins models, obtaining thereby the family of ARIMA (p, d, q) processes.

We formally define an ARIMA (p, d, q) process, where p and q are integers and d is real, to be a stochastic process $\{y_t\}$ which may be represented as $\phi(B)\nabla^d y_t = \theta(B) a_t$, where ∇^d is the fractional-differencing operator defined in (2·1), $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$ $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$ are polynomials in the backward-shift operator B, and $\{a_t\}$ is a white noise process.

The reason for choosing this family of processes for modelling purposes is that the effect of the d parameter on distant observations decays hyperbolically as the lag increases, while the effects of the ϕ and θ parameters decay exponentially. Thus d may be chosen to describe the high-lag correlation structure of a time series while the ϕ and θ parameters are chosen to describe the low-lag correlation structure. Indeed the long-term behaviour of an ARIMA (p, d, q) process may be expected to be similar to that of an ARIMA $(0, d, 0)$ process with the same value of d, since for very distant observations the effects of the ϕ and θ parameters will be negligible. Theorem 2 shows that this is indeed so.

THEOREM 2. *Let $\{y_t\}$ be an* ARIMA (p, d, q) *process. Then*

(1) *$\{y_t\}$ is stationary if $d < \frac{1}{2}$ and all the roots of the equation $\phi(z) = 0$ lie outside the unit circle*;

(2) $\{y_t\}$ *is invertible if* $d > -\frac{1}{2}$ *and all the roots of the equation* $\theta(z) = 0$ *lie outside the unit circle.*

If $\{y_t\}$ *is stationary and invertible, with spectral density* $s(\omega)$ *and correlation function* ρ_k, *then*

(3) $\lim \omega^{2d} s(\omega)$ *exists, as* $\omega \to 0$, *and is finite:*

(4) $\lim k^{1-2d} \rho_k$ *exists, as* $k \to \infty$, *and is finite.*

Proof. (1) Writing $y_t = \psi(B)\, a_t$, we have $\psi(z) = (1-z)^{-d}\, \theta(z)/\phi(z)$. Now the power series expansion of $(1-z)^{-d}$ converges for all $|z| \leqslant 1$ when $d < \frac{1}{2}$, that of $\theta(z)$ converges for all z and θ_i since $\theta(z)$ is a polynomial, and that of $1/\phi(z)$ converges for all $|z| \leqslant 1$ when all the roots of the equation $\phi(z) = 0$ lie outside the circle $|z| = 1$. Thus when all these conditions are satisfied, the power series expansion of $\psi(z)$ converges for all $|z| \leqslant 1$ and so $\{y_t\}$ is stationary.

(2) The proof is similar to (1) except that conditions are required on the convergence of $\pi(z) = (1-z)^d\, \phi(z)/\theta(z)$.

(3) We have

$$s(\omega) = \frac{\theta(e^{i\omega})\, \theta(e^{-i\omega})}{\phi(e^{i\omega})\, \phi(e^{-i\omega})} \{(1-e^{i\omega})(1-e^{-i\omega})\}^{-d} \sim \left(\frac{1-\theta_1-\ldots-\theta_q}{1-\phi_1-\ldots-\phi_p}\right)^2 (2\sin\tfrac{1}{2}\omega)^{-2d}$$

as $\omega \to 0$, which gives the required result.

(4) Let $x_t = \{\theta(B)\}^{-1}\phi(B)\, y_t$, so that $\nabla^d x_t = a_t$, that is $x_t \sim$ ARIMA $(0, d, 0)$; let $u_t = \nabla^d y_t$, so that $\phi(B)\, u_t = \theta(B)\, a_t$, that is $u_t \sim$ ARIMA $(p, 0, q)$. Then the covariances γ_j^u of $\{u_t\}$ satisfy a pth order difference equation (Box & Jenkins, 1976, p. 75), and we may write

$$\gamma_j^u = \sum_{l=1}^{p} \alpha_l\, \varepsilon_l^{j-q} \quad (j > q), \tag{4·1}$$

for some constants α_l, ε_l with, by stationarity, $|\varepsilon_l| < 1$ for all l. Now in an obvious notation

$$\gamma_k^y = \sum_{j=-\infty}^{\infty} \gamma_j^u \gamma_{k-j}^x,$$

and using (4·1) and (3·2) to substitute for γ_j^u and γ_j^x we obtain after some algebra

$$\gamma_k^y = \sum_{j=-q}^{q} \gamma_j^u \gamma_{k-j}^x + \gamma_{k+q}^x \sum_{l=1}^{p} \alpha_l \{F(1, d-q-k; 1-d-q-k;\, \varepsilon_l) - 1\}$$
$$+ \gamma_{k-q}^x \sum_{l=1}^{p} \alpha_l \{F(1, k-q+d;\, k-q+1-d;\, \varepsilon_l) - 1\}. \tag{4·2}$$

As $k \to \infty$, the hypergeometric functions $F(\,.\,)$ in (4·2) all tend to $(1-\varepsilon_l)^{-1}$ and we find that

$$\lim_{k\to\infty} \gamma_k^y/\gamma_k^x = \sum_{j=-q}^{q} \gamma_j^u + 2 \sum_{l=1}^{p} \alpha_l\, \varepsilon_l/(1-\varepsilon_l),$$

which is a finite constant; the result follows.

5. Two-parameter ARIMA (p, d, q) processes

In practice ARIMA (p, d, q) processes are likely to be of most interest for small values of p and q, so we now consider in more detail the simplest such processes, the ARIMA $(1, d, 0)$ and ARIMA $(0, d, 1)$.

The ARIMA $(1, d, 0)$ process is defined by $(1-\phi B)\nabla^d y_t = a_t$, where $\{a_t\}$ is a white noise process. We write $x_t = (1-\phi B)y_t$, so that $\nabla^d x_t = a_t$. Thus $\{y_t\}$ is a first-order autoregression with disturbances generated by an ARIMA $(0, d, 0)$ process. We expect $\{y_t\}$ to show similar long-term behaviour to $\{x_t\}$, but its short-term behaviour will depend on the value of ϕ. To ensure stationarity and invertibility of $\{y_t\}$ we assume $|\phi| < 1$ and $|d| < \frac{1}{2}$. The correlations of $\{y_t\}$ are most conveniently expressed in terms of the hypergeometric function

$$F(a,b;\, c;\, z) = 1 + \frac{ab}{c\,.\,1}z + \frac{a(a+1)\,b(b+1)}{c(c+1)\,.\,1\,.\,2}z^2 + \ldots.$$

From Theorem 2 we can deduce the following properties of $\{y_t\}$.

Lemma 1. *Let $\{y_t\}$ be a stationary invertible* ARIMA $(1, d, 0)$ *process* $(1-\phi B)\nabla^d y_t = a_t$; *let* $x_t = (1-\phi B)y_t$.

(a) *The ψ and π weights of the infinite moving-average and autoregressive representations of $\{y_t\}$ are given by*

$$\psi_k = \frac{(k+d-1)!}{k!\,(d-1)!}F(1, -k;\, 1-d-k;\, \phi) \sim \frac{k^{d-1}}{(1-\phi)\,(d-1)!},$$

$$\pi_k = \frac{(k-d-2)!}{(k-1)!\,(-d-1)!}\{1-\phi-(1+d)/k\} \sim \frac{(1-\phi)}{(-d-1)!}k^{-d-1},$$

as $k \to \infty$.

(b) *The spectral density of $\{y_t\}$ is*

$$s(\omega) = \frac{\{2\sin(\frac{1}{2}\omega)\}^{-2d}}{1+\phi^2-2\phi\cos\omega} \sim \frac{\omega^{-2d}}{(1-\phi)^2}$$

as $\omega \to 0$.

(c) *Let $\gamma_k^y, \rho_k^y, \gamma_k^x, \rho_k^x$ be the covariances and correlations of $\{y_t\}$ and $\{x_t\}$ respectively, the latter being given by Theorem* 1; *then*

$$\gamma_k^y = \gamma_k^x\{F(1, d+k;\, 1-d+k;\, \phi) + F(1, d-k;\, 1-d-k;\, \phi) - 1\}/(1-\phi^2),$$

$$\rho_k^y = \rho_k^x\frac{\{F(1, d+k;\, 1-d+k;\, \phi) + F(1, d-k;\, 1-d-k;\, \phi) - 1\}}{(1-\phi)\,F(1, 1+d;\, 1-d;\, \phi)}.$$

In particular,

$$\gamma_0^y = \frac{(-2d)!}{\{(-d)!\}^2}\,\frac{F(1, 1+d;\, 1-d;\, \phi)}{(1+\phi)}, \quad \rho_1^y = \frac{(1+\phi^2)\,F(1, d;\, 1-d;\, \phi) - 1}{\phi\{2F(1, d;\, 1-d;\, \phi) - 1\}}. \tag{5·1}$$

As $k \to \infty$,

$$\rho_k^y \sim \frac{(-d)!}{(d-1)!}\,\frac{(1+\phi)}{(1-\phi)^2}\,\frac{k^{2d-1}}{F(1, 1+d;\, 1-d;\, \phi)}.$$

For large k the correlation function decays hyperbolically. As ϕ varies, the lag-one correlation ρ_1^y can take any value in $(-1, +1)$ for any value of d in $[-\frac{1}{2}, \frac{1}{2})$.

The partial correlations ϕ_{kk} of $\{y_t\}$ have a complicated form and seem of little use in process identification. The partial correlation function jumps sharply between lags 1 and 2, and decays slowly, but not monotonically, to zero thereafter. Calculation of ϕ_{kk} for various values of d and ϕ suggests that $\phi_{kk} \sim d/k$ as $k \to \infty$; see Table 1.

Table 1. *Partial correlations at lag k of an* ARIMA $(1, d, 0)$ *process with* $d = 0{\cdot}2$ *for various values of* ϕ

k	$\phi = -0{\cdot}5$	$\phi = 0{\cdot}0$	$\phi = 0{\cdot}1$	$\phi = 0{\cdot}5$	$\phi = 0{\cdot}9$
1	−0·324	0·250	0·352	0·711	0·968
2	0·188	0·111	0·093	0·004	−0·145
3	0·095	0·071	0·065	0·032	−0·043
4	0·064	0·053	0·049	0·031	−0·018
5	0·048	0·042	0·040	0·028	−0·007
10	0·022	0·020	0·020	0·017	0·004
20	0·010	0·010	0·010	0·009	0·005
100	0·002	0·002	0·002	0·002	0·002

Example. Consider the ARIMA $(1, d, 0)$ process with $d = 0{\cdot}2$, $\phi = 0{\cdot}5$. It has $\rho_1 = 0{\cdot}711$, $0 < \phi_{kk} < 0{\cdot}04$ for $k > 1$. So by looking at its partial correlations we might identify it as an ARIMA $(1, 0, 0)$ process with $\phi = 0{\cdot}711$. But the partial correlations of the ARIMA $(1, d, 0)$ process, though small, are all positive and the correlation functions of the two processes differ markedly after the first few lags; see Table 2.

Table 2. *Correlations of an* ARIMA $(1, d, 0)$ *process with* $d = 0{\cdot}2$, $\phi = 0{\cdot}5$ *and an* AR (1) *process with* $\phi = 0{\cdot}711$

k	ρ_k ARIMA	ρ_k AR	k	ρ_k ARIMA	ρ_k AR
1	0·711	0·711	7	0·183	0·092
2	0·507	0·505	8	0·166	0·065
3	0·378	0·359	9	0·152	0·046
4	0·296	0·255	10	0·141	0·033
5	0·243	0·181	15	0·109	0·001
6	0·208	0·129	20	0·091	0·000

The ARIMA $(0, d, 1)$ process may be thought of as a first-order moving average of fractionally differenced white noise. The process is defined by $\nabla^d y_t = (1 - \theta B) a_t$, where $\{a_t\}$ is white noise; it is stationary and invertible for $|\theta| < 1$, $|d| < \frac{1}{2}$. From Theorem 2 we obtain the following results.

LEMMA 2. *Let* $\{y_t\}$ *Be a stationary invertible* ARIMA $(0, d, 1)$ *process; let* $\nabla^d x_t = a_t$, *so that* $y_t = (1 - \theta B) x_t$; *write* $\gamma_k^y, \rho_k^y, \gamma_k^x, \rho_k^x$ *for the covariances and correlations of* $\{y_t\}$, $\{x_t\}$ *respectively.*

(a) *The* ψ *and* π *weights of* $\{y_t\}$ *are equal to the* π *and* ψ *weights respectively of an* ARIMA $(1, -d, 0)$ *process with autoregressive parameter* θ.

(b) *The spectral density of* $\{y_t\}$ *is*

$$s(\omega) = (1 + \theta^2 - 2\theta \cos \omega) \{2 \sin(\tfrac{1}{2}\omega)\}^{-2d} \sim (1 - \theta)^2 \omega^{-2d}$$

as $\omega \to 0$.

(c) *The covariance function of* $\{y_t\}$ *is*

$$\gamma_k^y = \gamma_k^x \frac{(1-\theta)^2 k^2 - (1-d)\{(1-d)(1+\theta^2) - 2\theta d\}}{k^2 - (1-d)^2}$$

and the correlation function of $\{y_t\}$ *is*

$$\rho_k^y = \rho_k^x \frac{ak^2 - (1-d)^2}{k^2 - (1-d)^2},$$

where $a = (1-\theta)^2/\{1+\theta^2 - 2\theta d/(1-d)\}$. *In particular*

$$\gamma_0^y = \{1+\theta^2 - 2\theta d/(1-d)\}(-2d)!/\{(-d)!\}^2,$$

$$\rho_1^y = \frac{a-(1-d)^2}{(1-d)(2-d)} = \frac{(1+\theta^2)d(2-d) - 2\theta(1-d+d^2)}{(1-d)(2-d)\{1+\theta^2 - 2\theta d/(1-d)\}}. \qquad (5{\cdot}2)$$

As $k \to \infty$,

$$\rho_k^y \sim \frac{(-d)!}{(d-1)!} a k^{2d-1}.$$

For large k the correlation function of $\{y_t\}$ decays hyperbolically, while for given d the lag-one correlation ρ_1^y can take a range of values from a minimum of $-(1-d)/(2-d)$ at $\theta = +1$ to a maximum of $(1+d)/(2-d)$ at $\theta = -1$. For given values of d and ρ_1^y between these limits, the corresponding value of θ may be found by solving (5·2), which is quadratic in θ and thus easier to solve than the corresponding equation (5·1) for an ARIMA $(1, d, 0)$ process.

Example. Table 3 gives the correlations of an ARIMA $(1, d, 0)$ process and an ARIMA $(0, d, 1)$ process chosen to have the same values of d and ρ_1^y. The two correlation functions behave similarly at high lags, but the ARIMA $(0, d, 1)$ correlation function drops more sharply between lags 1 and 2. The nonzero value of d means that the ARIMA $(0, d, 1)$ correlation function decays steadily towards zero from lag 2 onwards, rather than the second and higher correlations being zero as they would for an ARIMA $(0, 0, 1)$ process.

Table 3. *Correlations of an* ARIMA $(1, d, 0)$ *process with* $d = 0{\cdot}2$, $\phi = 0{\cdot}366$ *and an* ARIMA $(0, d, 1)$ *process with* $d = 0{\cdot}2$, $\theta = -0{\cdot}508$

k	ρ_k ARIMA $(1, d, 0)$	ρ_k ARIMA $(0, d, 1)$
1	0·600	0·600
2	0·384	0·267
3	0·273	0·202
4	0·213	0·168
5	0·178	0·146
10	0·111	0·096
20	0·073	0·063
100	0·028	0·024

6. Conclusion

The family of ARIMA (p, d, q) processes defined in § 5 should prove useful in many fields of time series analysis. Compared with existing models of long-term persistence, the ARIMA family is much more flexible in the simultaneous modelling of long-term and short-term behaviour of a stochastic process. Since the family is a generalization of the well-known Box–Jenkins models, the existing methodology may now be applied to time series exhibiting long-term persistence or antipersistence.

An outline of a procedure for identification and estimation of an ARIMA (p, d, q) model is as follows. We write the ARIMA (p, d, q) model as $\phi(B)\nabla^d y_t = \theta(B)a_t$; we define $u_t = \nabla^d y_t$, so that $\{u_t\}$ is an ARIMA $(p, 0, q)$ process, and $x_t = \{\theta(B)\}^{-1}\phi(B)\, y_t$, so that $\{x_t\}$ is an ARIMA $(0, d, 0)$ process.

(1) Estimate d in the ARIMA $(0, d, 0)$ model $\nabla^d y_t = a_t$.
(2) Define $u_t = \nabla^d y_t$.
(3) Using the Box–Jenkins modelling procedure, identify and estimate the ϕ and θ parameters in the ARIMA $(p, 0, q)$ model $\phi(B)\, u_t = \theta(B)\, a_t$.
(4) Define $x_t = \{\theta(B)\}^{-1}\phi(B)\, y_t$.
(5) Estimate d in the ARIMA $(0, d, 0)$ model $\nabla^d x_t = a_t$.
(6) Check for convergence of the d, ϕ and θ parameters; if not converged go to step 2.

The estimate of d in steps 1 and 5 may be the rescaled range exponent, or R/S exponent, used as a measure of long-term persistence by Mandelbrot & Wallis (1969) and Mandelbrot & Taqqu (1979); alternatively, a maximum likelihood estimate of d may be obtained by the methods of McLeod & Hipel (1978, p. 497), if we use the correlation function of an ARIMA $(0, d, 0)$ process in their equation (38).

One of the main reasons why Mandelbrot & Wallis (1969) introduced fractional Gaussian noise was to enable computer generation of synthetic time series exhibiting long-term persistance for use in hydrological simulation studies. The ARIMA $(0, d, 0)$ process may also be applied in this fashion; an efficient method of generation may be based on the partial correlations of the process, since these take the simple form (3·4). The ARIMA $(1, d, 0)$ and ARIMA $(0, d, 1)$ processes are convenient for generating long-term persistent series having specific values of the lag-one correlation.

The processes so far considered have generalized the Box–Jenkins ARIMA (p, d, q) processes by allowing d to take fractional values; we may also consider the possibility of allowing p and q to take fractional values. In general this is not meaningful, since p and q are degrees of polynomials $\phi(B)$ and $\theta(B)$. However, in the particular case of a polynomial with equal roots, say $\phi(B) = (1-\phi B)^p$, a generalization of p to fractional values may be entertained, leading to the equal-root autoregressive process of fractional order $(1-\phi B)^p\, y_t = a_t$. This process and its applicability to hydrological modelling are described by Spolia, Chander & O'Connor (1980); the covariance function of the process is, as $k \to \infty$,

$$\gamma_k = \phi^k \frac{(k+p-1)!}{k!\,(p-1)!} F(p, k+p;\, k+1;\, \phi^2) \sim \frac{(1-\phi^2)^{-p}}{(p-1)!} \phi^k k^{p-1}.$$

Finally we mention two other processes involving fractional differencing which may prove useful in applications. The fractional equal-root integrated moving-average process is defined by $\nabla^q y_t = (1-\theta B)^q a_t$, $|q| < \frac{1}{2}$, $|\theta| < 1$; as a forecasting model it corresponds to 'fractional order multiple exponential smoothing'. The process $(1-2\phi B+B^2)^d\, y_t = a_t$, $|d| < \frac{1}{2}$, $|\phi| < 1$, exhibits both long-term persistence and quasiperiodic behaviour; its correlation function resembles a hyperbolically damped sine wave.

REFERENCES

ADELMAN, I. (1965). Long cycles—fact or artifact? *Am. Econ. Rev.* **60**, 444–63.

BOX, G. E. P. & JENKINS, G. M. (1976). *Time Series Analysis Forecasting and Control*, 2nd edition. San Francisco: Holden-Day.

CHATFIELD, C. (1979). Inverse autocorrelations. *J. R. Statist. Soc.* A **142**, 363–77.

COX, D. R. (1977). Contribution to discussion of paper by A. J. Lawrance and N. T. Kottegoda. *J. R. Statist. Soc.* A **140**, 34.

GRADSHTEYN, I. S. & RYZHIK, I. W. (1965). *Table of Integrals, Series and Products*. New York: Academic Press.

GRANGER, C. W. J. (1966). The typical spectral shape of an economic variable. *Econometrica* **34**, 150–61.

GRANGER, C. W. J. (1978). New classes of time series models. *Statistician* **27**, 237–53.

HIPEL, K. W. & MCLEOD, A. I. (1978). Preservation of the rescaled adjusted range. 2. Simulation studies using Box–Jenkins models. *Water Resour. Res.* **14**, 509–16.

LAWRANCE, A. J. & KOTTEGODA, N. T. (1977). Stochastic modelling of riverflow time series (with discussion). *J. R. Statist. Soc.* A **140**, 1–47.

MANDELBROT, B. B. (1965). Une classe de processus stochastiques homothétiques à soi; application à la loi climatologique de H. E. Hurst. *Comptes Rendus Acad. Sci. Paris* **260**, 3274–7.

MANDELBROT, B. B. (1967). Some noises with $1/f$ spectrum, a bridge between direct current and white noise. *I.E.E.E. Trans. Info. Theory* **13**, 289–98.

MANDELBROT, B. B. (1977). *Fractals: Form, Chance and Dimension*. San Francisco: Freeman.

MANDELBROT, B. B. & TAQQU, M. S. (1979). Robust R/S analysis of long run serial correlation. *Bull. Int. Statist. Inst.* **48**. To appear.

MANDELBROT, B. B. & VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *S.I.A.M. Rev.* **10**, 422–37.

MANDELBROT, B. B. & WALLIS, J. R. (1969). Computer experiments with fractional Gaussian noises. *Water Resour. Res.* **5**, 228–67.

MATALAS, N. C. & WALLIS, J. R. (1971). Statistical properties of multivariate fractional noise processes. *Water Resour. Res.* **7**, 1460–8.

MCLEOD, A. I. & HIPEL, K. W. (1978). Preservation of the rescaled adjusted range. 1. A reassessment of the Hurst phenomenon. *Water Resour. Res.* **14**, 491–508.

SPOLIA, S. K., CHANDER, S. & O'CONNOR, K. M. (1980). An autocorrelation approach for parameter estimation of fractional order equal-root autoregressive models using hypergeometric functions. *J. Hydrol.* **47**, 1–18.

WALLIS, J. R. & MATALAS, N. C. (1970). Small sample properties of H and K, estimators of the Hurst coefficient h. *Water Resour. Res.* **6**, 1583–94.

[*Received May* 1980. *Revised September* 1980]