2. Simulation studies using Box- Jenkins models', *Water Resource Research* **14**, 509–516.

Hosking, J. (1981), 'Fractional differencing', *Biometrika* **68**, 165–76.

Hurst, H. (1951), 'Long-term storage of reservoirs', *Transactions of the American Society of Civil Engineers* **116**.

Lo, A. W. (1991), 'Long term memory in stock market prices', *Econometrica* **59**, 1279–1313.

Mandelbrot, B. & Van Ness, J. (1968), 'Fractional Brownian motion, fractional noise, and applications', *SIAM Review* **10**.

Mandelbrot, B. & Wallis, J. (1969), 'Computer experiments with fractional Gaussian noises. Part 3, mathematical appendix', *Water Resources Research* **3**(1), 260–267.

McLeod, A. & Hipel, K. (1978), 'Preservation of the rescaled adjusted range. 1. A reassessment of the Hurst phenomenon', *Water Resource Research* **14**, 491–508.

Moody, J. & Wu, L. (1994), Statistical analysis and forecasting of high frequency foreign exchange rates, *in* 'International workshop on Neural Networks in the Capital Markets', Pasadena, California.

Moody, J. & Wu, L. (1995*a*), Price behavior and Hurst exponents of tick-by-tick interbank foreign exchange rates, *in* 'Conference on Computational Intelligence for Financial Engineering', New York city.

Moody, J. & Wu, L. (1995*b*), Statistical analysis and forecasting of high frequency foreign exchange rates, *in* 'First international conference on High Frequency data in Finance', Zurich, Switzerland.

Müller, U., Dacorogna, M. & Pictet, O. (1995), The error of statistical volatility of intra-daily quoted price changes observed over a time interval, Technical Report UAM. 1995-07-31, Olsen & Associates, Zurich, Switzerland.

Müller, U., Dacorogna, M., Olsen, R., Pictet, O., Schwarz, M. & Morgenegg, C. (1990), 'Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis', *Jounal of Banking and Finance* **14**, 1189–1208.

Peters, E. (1989), 'Fractal structure in the capital markets', *Financial Analysts Journal* pp. 32–37.

Wallis, J. & Matalas, N. (1970), 'Small sample properties of H and K, estimators of the Hurst coefficient H', *Water Resource Research* **6**, 1583–94.

standard deviation to correct this bias when the observed time scale is large enough. When the time scale is small, however, the $R/\tilde{S}$ statistic introduces new estimation errors due to negative biases in $\tilde{S}$. This effect also results in downward errors in estimated Hurst exponents. Our proposed R/S^* statistic overcomes deficiencies in both R/S and $R/\tilde{S}$ correcting for short term dependencies in the time series without introducing additional biases on short time scales N.

6.2. The Tick-by-Tick DEM/USD Series

As demonstrated in (Moody & Wu (1994) Moody & Wu (1995*a*) and Moody & Wu (1995*b*)), there exist very significant one or two tick anti-correlations in the returns of the DEM/USD series. When analyzing the price behavior and forecasting price changes on longer time scales, such short-term anti-correlations should be removed. We have demonstrated in this paper that not considering these effects results in completely different conclusions about the behavior of the series as measured by the Hurst exponents on time scales of 10 to 100 ticks.

As shown in (Moody & Wu (1995*b*)), simply down-sampling the price series cannot removed this short-term anti-correlation. However, our proposed $[R/S^*](N)$ analysis confirms our previous results obtained by short term block averaging that the DEM/USD series is actually mildly trending on time scales of 10 to 100 ticks, and that the suggested mean-reversion in the R/S and $R/\tilde{S}$ analyses on these time scales is spurious.

7. Acknowledgements

We thank Steve Rehfuss for valuable comments and discussions. We gratefully acknowledge support for this work from ARPA and ONR under grant N00014-92-J-4062 and NSF under grant CDA-9309728.

8. References

Ambrose, B., Ancel, E. & Griffiths, M. (1993), 'Fractal structure in the capital markets revisited', *Financial Economic Review* **30**, 685–704.

Andrews, D. (1991), 'Heteroskedasticity and autocorrelation consistent covariance matrix estimation', *Econometrica* **59**, 817–858.

Feder, J. (1988), *Fractals*, Plenum Press, New York.

Granger, C. & Joyeux, R. (1980), 'An introduction to long-memory time series models and fractional differencing', *Journal of Time Series Analysis* **1**, 15–29.

Guillaume, D. (1994), On the trend-following behavior of intra-daily foreign exchange market and its relationship with the volatility, Unpublished manuscript, Center for Economic Studies, Catholic University of Leuven.

Hipel, K. & McLeod, A. (1978), 'Preservation of the rescaled adjusted range.

mean-reversion on these time scales in the $[R/S](N)$ curves is actually due to high frequency oscillations on time scales of a few ticks.

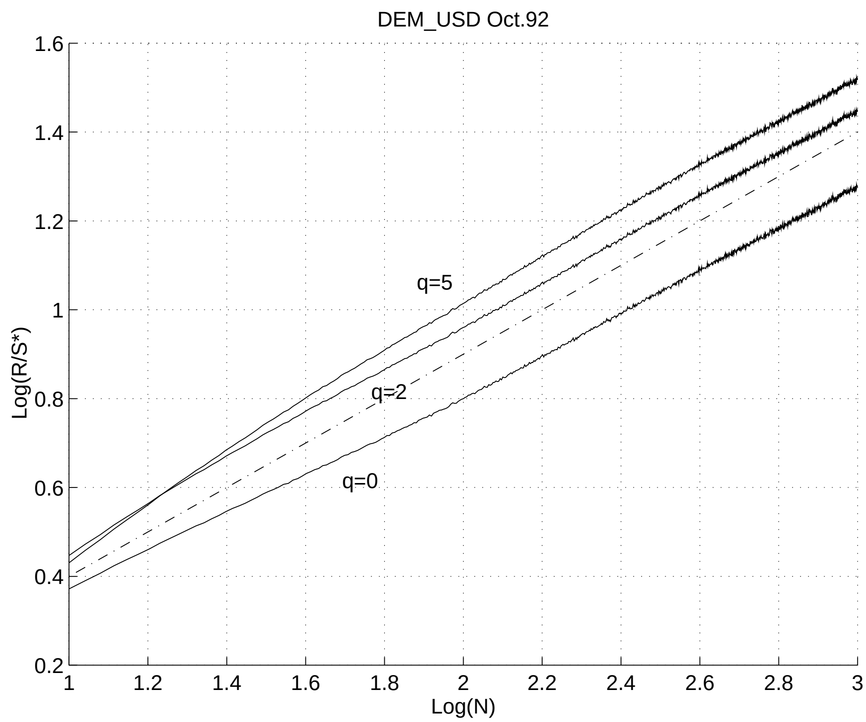


Figure 8: $[R/S^*](N)$ for the DEM/USD bid returns for October 1992 using lag parameters $q = \{0, 2, 5\}$. Comparing the results for $q = 0$ to the $[R/S](N)$ curves for the unscrambled FX data in Figures 1 and 2 (a), we see slightly less apparent mean reversion at short time scales N. This reduction in apparent mean reversion is due to the unbiasedness of S^* relative to S. The results for $q = \{2, 5\}$ are qualitatively similar to the block averaged results for blocks of 4 and 8 ticks shown in Figures 2 (c) and (d). This supports our conclusion that the apparent mean reversion on short time scales in the DEM/USD series is actually due to the high frequency oscillations, and that when these are removed, the series is actually slightly trending.

6. Concluding Remarks and Discussions

6.1. The R/S, $R/\tilde{S}$, and R/S^* Statistics and Hurst Exponents

Due to the use of a biased estimate of the standard deviation, the R/S and $R/\tilde{S}$ statistics are biased upward on short time scales, resulting in downward errors in estimated Hurst exponents. When short-term dependencies are present, the estimated range in R/S analysis may be biased. The $R/\tilde{S}$ statistic adds autocovariance to the

the proposed unbiased rescaling factor with weighted covariances up to lag q is:

$$S^*(N, t_0, q) = \left\{ \left[1 + 2\sum_{j=1}^{q} w_j(q) \frac{N-j}{N^2} \right] \widehat{\sigma}^2(N, t_0) + \frac{2}{N} \sum_{j=1}^{q} w_j(q) \sum_{t=t_0+j}^{t_0+N} (r_t - m)(r_{t-j} - m) \right\}^{1/2} , \qquad (16)$$

where $w_j(q)$ is the weighting function as defined by Lo ($w_j(q) = 1 - \frac{j}{q+1}$). This weighting function yields a positive S^{*2}, provided that $q < N$. It is trivial to show that the estimates of the autocovariances in (16) have zero mean bias. When $q = 0$, S^* reduces to the unbiased standard deviation $\widehat{\sigma}$.

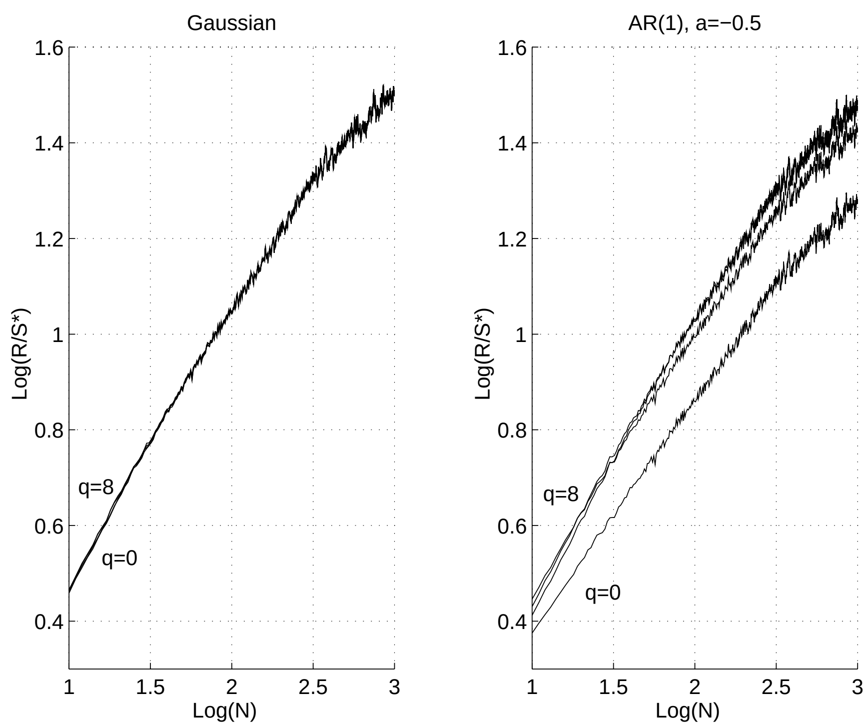
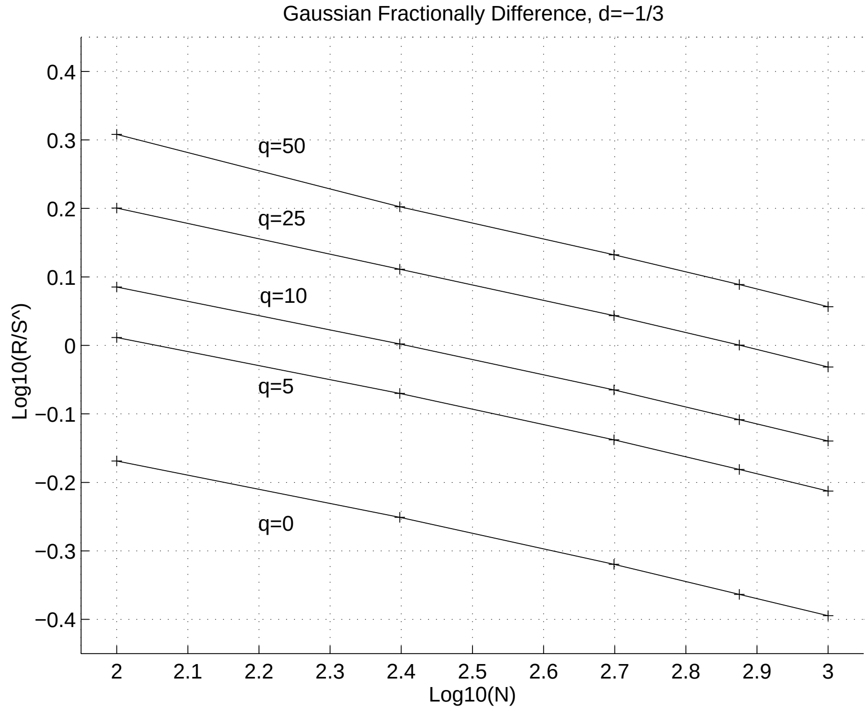


Figure 7: $[R/S^*](N)$ for a gaussian i.i.d. returns process (left panel) and an AR(1) returns process with negative coefficient (right panel) using different lag parameters $q = \{0, 2, 4, 8\}$. Note that unlike the results for $[R/\tilde{S}](N)$ in Figure 4, the mean bias in R for the AR(1) process is corrected with $q = 8$ without inducing distortions for short time scales N. Note also, that unlike the results for $[R/\tilde{S}](N)$, the results for the gaussian i.i.d. process are independent of q.

Figures 7 and 8 present empirical results for the proposed $[R/S^*](N)$ statistic. The results for simulated gaussian i.i.d. and AR(1) returns processes shown in Figure 7 confirm the efficacy of our proposed $[R/S^*](N)$ analysis. The results for the high frequency DEM/USD FX series in Figure 8 support our conclusions obtained by block averaging in Figure 2 that: (1) the DEM/USD series is actually trending, rather than mean-reverting on short time scales (10 to 100 ticks), and (2) the spuriously-observed



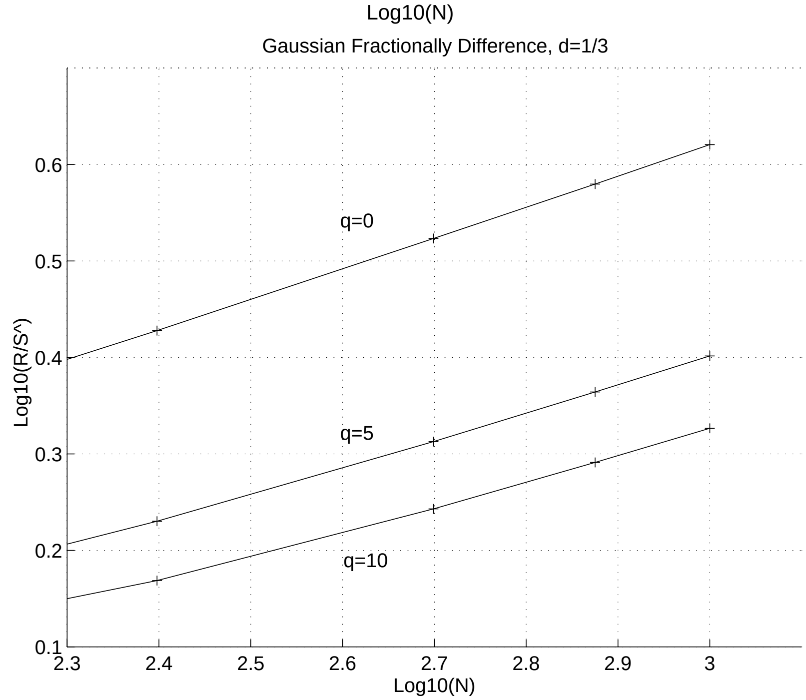


Figure 6: Lo's $R/\tilde{S}$ curves for gaussian fractional differenced processes. The curves are constructed based on Lo's own simulation results (see Table VIa and Table VIb in (Lo (1991))). All the $R/\tilde{S}$ curves have similar slopes in all time scales with different time lags. For the process with $d = -1/3$, all the $R/\tilde{S}$ slopes are negative. This is incompatible with the allowed scaling laws for stochastic processes.

respectively. We have:

$$\frac{C}{S^2} = \frac{2\sum_{j=1}^{q} w_j(q) \sum_{t=t_0+j}^{t_0+N} (r_t - m)(r_{t-j} - m)}{\sum_{t=t_0+1}^{t_0+N} (r_t - m)^2} \quad . \tag{14}$$

If there is no short-term dependence and $\frac{C}{S^2} = 0$, Eq.(13) reduces to that for the R/S analysis. If N is small, $\frac{C}{S^2}$ will change with N. The slope of $log\left(\frac{R}{S}\right)$ against $log(N)$ in Eq.(13) will be modified. If N is large enough, $\frac{C}{S^2}$ will not depend on the value of N. The effect of $\frac{1}{2}log\left(1 + \frac{C}{S^2}\right)$ in Eq.(14) is the same as that of b. Its existence will shift the R/S curve vertically, but will not change the slope of the curve. The above explanation is completely consistent with the empirical results shown in Figures 3, 4, 5 and Table 1. We therefore confirm that the $R/\tilde{S}$ analysis and the R/S analysis have the same Hurst exponents when the time scale N is large.

In Lo's own simulation results in (Lo (1991)), for example, Table Vb for an AR(1) process and Table VIb for a gaussian fractional differenced process, we also find similar evidence. For the AR(1) process, when $N \geq 250$, the $R/\tilde{S}$ curves with q=0,5,10 have very close slopes. For the gaussian fractional differenced process, all $R/\tilde{S}$ curves have similar slopes on all time scales with q=0,5,10,25 and 50. Figure 6 depicts the $R/\tilde{S}$ curves of Lo's simulation results. On the other hand, the upper panel of Figure 6 shows that Lo's $[R/\tilde{S}](N)$ may have a negative slope, which is not theoretically justifiable using the standard definition of Hurst exponents, and is incompatible with the scaling laws for stochastic processes (Feder (1988)).

In summary, Lo tests the random walk hypothesis directly based on the value of the R/S or $R/\tilde{S}$ statistic, while Hurst and Mandelbrot do so by comparing the slope of $[R/S](N)$ curve to 0.5. Unfortunately, biases in the definitions of R, S and $\tilde{S}$ can lead to errors in the estimates of $[R/S](N)$, $[R/\tilde{S}](N)$, and H for short time scales N or when short term dependencies are present in the series under study. Under standard interpretations of $[R/S](N)$, $[R/\tilde{S}](N)$, and H, these errors can lead to misleading and sometimes inconsistent results.

5. Rescaled Range Analysis with Unbiased S*

To address the problems with the statistics $[R/S](N)$ and $[R/\tilde{S}](N)$ described above, we propose an unbiased rescaling factor S^* that corrects for mean biases in the range R due to short-term dependencies without inducing the distortions on short time scales that S and Lo's $\tilde{S}$ do. Denoting the standard unbiased estimate of the variance as

$$\hat{\sigma}^2(N, t_0) = \frac{1}{N-1} \sum_{t=t_0+1}^{t_0+N} (r_t - m)^2 \quad , \tag{15}$$

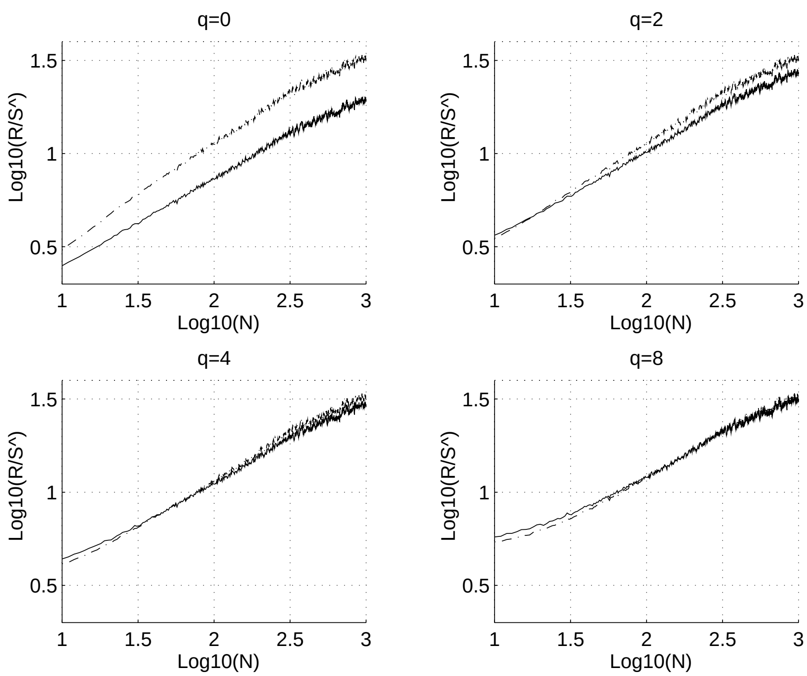


Figure 5: Comparison of the $R/\tilde{S}$ analysis for an AR(1) series (solid curve) to that for a gaussian series (dashed curve) for $q = 0, 2, 4, 8$. The bias of the range due to the short-term dependence in AR(1) series can clearly be seen in the curves with $q = 0, 2, 4$. When $q = 8$, the bias is corrected and the $R/\tilde{S}$ curve of the AR(1) series overlaps that of gaussian series for long time scales N.

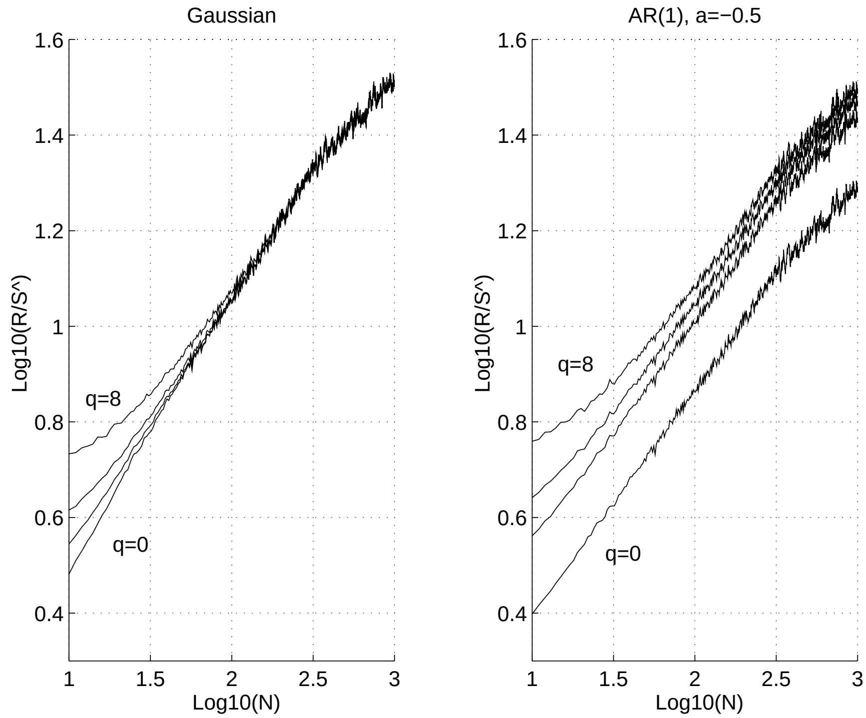


Figure 4: $R/\tilde{S}$ analysis for a gaussian i.i.d. returns process (left panel) and an AR(1) returns process with negative coefficient (right panel) using different lag parameters $q = 0, 2, 4, 8$. Note that the $R/\tilde{S}$ statistic introduces a new estimation error when the observed time scale N is small.

large time scales, $R/\tilde{S}$ curves for the AR(1) process shift upward for increasing q and overlap the $R/\tilde{S}$ curves for the i.i.d. process only for $q = 8$. This demonstrates that the bias of the range exists in the R/S statistic and that the $R/\tilde{S}$ statistic can correct this bias for large N with a proper choice of the lag parameter q.

The left panel of Figure 4 illustrates our second observation. Since there is no dependence in the time series, we expect that all the $R/\tilde{S}$ curves should be equivalent for different lag parameters q for all time scales N. However, as shown by Figure 4, the curves for different q are not the same when the time scale N is small. This effect is due to the negative bias in the second term of Eq.(10).

To explain our third observation, we first rewrite the $R/\tilde{S}$ analysis

$$log\left(\frac{R}{\tilde{S}}\right) = Hlog(N) + b \tag{12}$$

as

$$log\left(\frac{R}{S}\right) = Hlog(N) + b + \frac{1}{2}log\left(1 + \frac{C}{S^2}\right) \quad . \tag{13}$$

Where b is constant and S^2 and C are the first and the second terms of Eq.(10)

Table 1: Estimated Hurst exponents from the $R/\tilde{S}$ analysis for DEM/USD data in October 1992.

Lags	Time Scales	
q	10-100	100-1000
0	0.406	0.477
2	0.409	0.476
4	0.392	0.473
8	0.342	0.466

4. $R/\tilde{S}$ Analysis and Hurst Exponents

Table 1 lists the estimated Hurst exponents from the $R/\tilde{S}$ analysis over the time scales $10-100$ and $100-1000$. From the table, we can see that: (1) for the smaller time scales $N = 10 - 100$, the Hurst exponents decrease significantly as the lag q increases; (2) for the larger time scales $N = 100 - 1000$, the Hurst exponents decrease only slightly with increasing lag q; and (3) all Hurst exponents are less than 0.5.

By observing a series of empirical results in our studies, we have found the following. (1) There indeed exists an estimation bias in the range statistic R due to short-term dependencies in the series that shifts the standard rescaled range statistic R/S. When the time scale N is large, the modified $R/\tilde{S}$ statistic can correct this bias. (2) When the time scale N is small, the rescaling factor $\tilde{S}$ introduces some new errors in estimating both the rescaled range statistic $R/\tilde{S}$ and the Hurst exponents. (3) When the time scale N is large, the slopes of $R/\tilde{S}$ curves are independent of the lag q, even though they shift vertically. This means that the $R/\tilde{S}$ analysis (when $q > 0$) and the R/S analysis (when $q = 0$) yield the same Hurst exponents.

To investigate the above issues further, we have conducted the $R/\tilde{S}$ analysis for simulated gaussian random walk and AR(1) series.

To demonstrate our first observation, we compare the $R/\tilde{S}$ analysis for a simulated gaussian i.i.d. returns process and an AR(1) returns process with regression coefficient $\alpha = -0.5$. For sufficiently large N, we know that the estimated autocovariances for the i.i.d. process will be very small and fall inside the 95% significance band for non-zero lags. Similarly, the autocovariances for the AR(1) process $\alpha^j \sigma^2/(1-\alpha^2)$ decay exponentially with lag j, and their estimates for finite N fall within the 95% significance band for large enough j. Therefore, we expect that for the i.i.d. process, the $R/\tilde{S}$ curves for different q should be the same for large time scales (N) and that the $R/\tilde{S}$ curves for the AR(1) process for large N will approach those of the i.i.d. process with increasing q. These effects are illustrated by Figures 4 and 5, where for

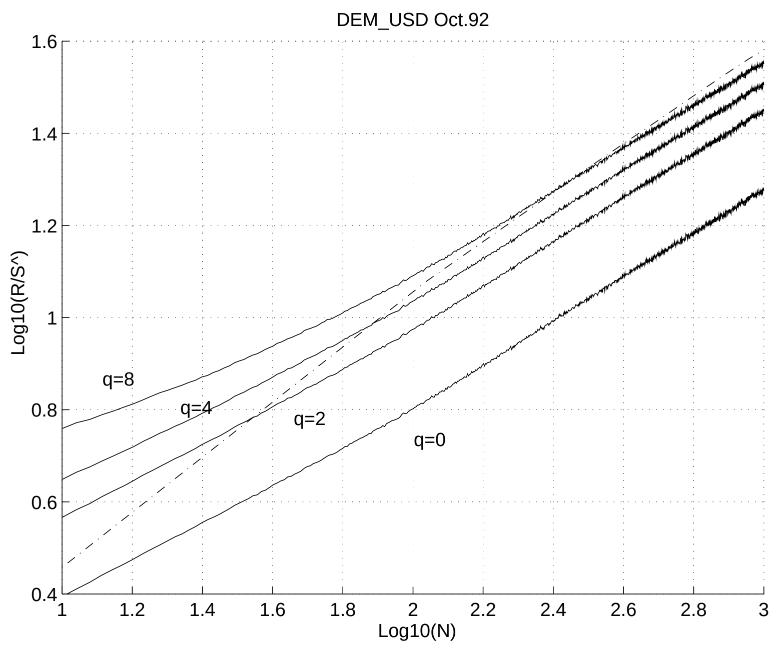


Figure 3: $R/\tilde{S}$ analysis for DEM/USD data in October 1992. Four solid curves correspond to the analysis for the same data but using different lag q for computing the autocovariances. When $q = 0$, the $R/\tilde{S}$ analysis reduces to the R/S analysis. The dashed straight line is with a slope of 0.5 and corresponds to a random walk. Note that when the time scale is large ($N = 100 - 1000$), all $R/\tilde{S}$ curves have very similar slopes and the slopes are all smaller than that for the random walk.

where $w_j(q)$ is defined as

$$w_j(q) = 1 - \frac{j}{q+1} \quad . \tag{11}$$

This weighting function always yields a positive $\tilde{S}$. The determination of q is rather complicated. Not being able to determine q by a simple closed-form expression, Lo discussed the effect of varying q. When q becomes large relative to N, the finite-sample distribution of the estimator can be radically different from its asymptotic limit. However, q cannot be chosen too small, since the autocovariances beyond lag q may be substantial and should be included in the weighted autocovariances. Therefore, the truncation lag must be chosen with some consideration of the data at hand.

The modified R/S analysis rescales the range R using $\tilde{S}$ instead of the standard deviation S, so we refer to it as $R/\tilde{S}$ *analysis*. As we shall demonstrate empirically in sections 3.3 and 4 Lo's rescaling factor $\tilde{S}$ has significant downward bias for small N, and this distorts both $R/\tilde{S}$ and the Hurst exponent H.

3.3. $R/\tilde{S}$ Analysis for Foreign Exchange Rates

In Section 2, we found that the interbank tick-by-tick foreign exchange price changes show mean-reverting behavior on time scales from several ticks to hundreds of ticks. However, when we consider the block average prices (averaged over a few ticks), the mean-reverting behavior on longer time scales is not significant, suggesting that the apparent mean-reverting behavior is not due to the fundamental nature of the price movements, but rather is just an artifact induced by the high frequency oscillations. In the following, we use $R/\tilde{S}$ analysis and try to get an answer.

We now consider Lo's modified $R/\tilde{S}$ analysis to see whether it can help confirm our conclusion above that the observed mean reverting behavior on intermediate time scales is actually due to the high frequency oscillations. Unfortunately, however, we find that the $R/\tilde{S}$ statistic introduces new problems and is not helpful in resolving this issue.

The $R/\tilde{S}$ analysis for the prices of DEM/USD exchange rates in October 1992 is plotted in Figure 3. The data used consists of 134,813 ticks. We take the price as the average of Bid and Ask quotes. The effect of the lag q is also depicted in the figure. Since the high frequency oscillations in the foreign exchange data (believed by some to be an inventory effect) are on very short time scales in tick time, we only use small q ($q = \{0, 2, 4, 8\}$) in our analysis. The case with $q = 0$ corresponds to the standard R/S analysis. From the figure, we see that (1) the $R/\tilde{S}$ curves shift upwards, compared to the R/S curve, and (2) the upward shift when the time scale N is small is significantly larger than that when N is large. This is due to the downward biases in $\tilde{S}$.

behavior.

3. Lo's Modified R/S Analysis

3.1. Mean Bias in Range Statistic

While studying long-term memory structures in stock prices using R/S analysis, Lo (1991) found that rejections of the null hypothesis (that the time series is a random walk) on long time scales can be erroneous and can be due instead to bias induced by short-term dependencies. He compared the asymptotic distributions of the rescaled ranges between an i.i.d random series and an AR(1) short-term dependent series. When the time scale N increases without bound, the normalized rescaled range of an i.i.d series converges to the range of a Brownian bridge on the unit interval,

$$\frac{R(N,t_0)}{\sqrt{N}} \mapsto B \qquad \text{i.i.d. Series} \tag{8}$$

However, for a short-term dependent AR(1) series with a regression coefficient α, the normalized rescaled range converges to

$$\frac{R(N,t_0)}{\sqrt{N}} \mapsto \xi B \qquad \text{AR(1) Series.} \tag{9}$$

The mean is biased by a factor of ξ, which for this special case is $\xi = \sqrt{(1+\alpha)/(1-\alpha)}$. The bias can be significant. For example, if $\alpha = -0.5$, the normalized rescaled range (9) is biased downward by a factor 0.577. Therefore, the short-term dependence will bias the estimation of the long-term rescaled range.

3.2. Modified R/S Analysis

To remove the effect of mean bias due to short-term dependencies, Lo (1991) proposed a modified R/S statistic. His motivation was that if r_t is subject to short-term dependence, the autocovariances of r_t will not be equal to zero, and the range R cannot simply be normalized by the standard deviation alone. The covariances should be considered also (Andrews (1991)). The rescaling term suggested by Lo, includes weighted covariances up to lag q and has the form:[b]

$$\tilde{S}(N,t_0,q) = \left\{ \frac{1}{N} \sum_{t=t_0+1}^{t_0+N} (r_t - m)^2 + \frac{2}{N} \sum_{j=1}^{q} w_j(q) \sum_{t=t_0+j}^{t_0+N} (r_t - m)(r_{t-j} - m) \right\}^{1/2} , \tag{10}$$

[b]Note that the $\tilde{S}$ is biased downward. The first term in (10) is biased by a factor $(N-1)/N$, while the second term has negative bias. This issue will be addressed empirically in sections 3.3 and 4 Section 5 presents an unbiased replacement S^*.

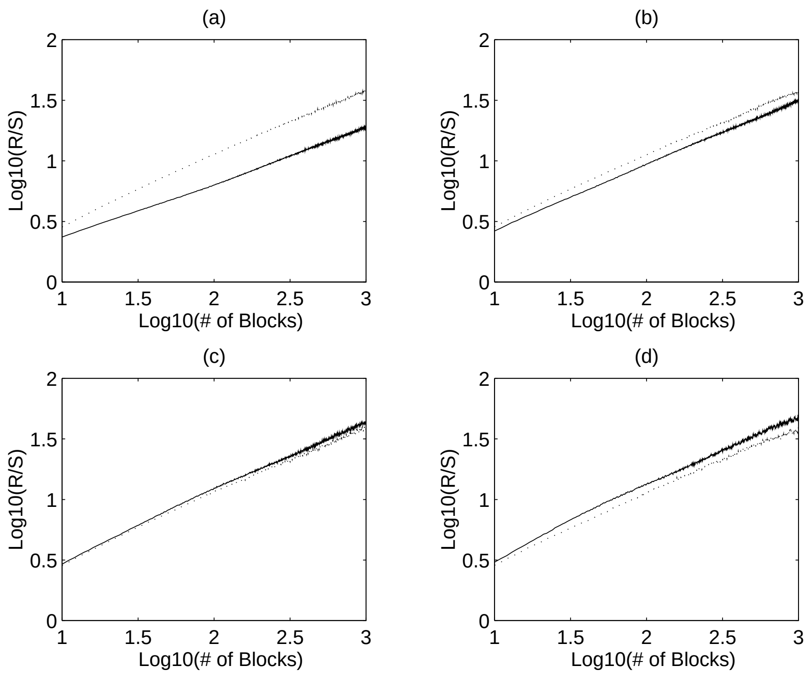


Figure 2: Rescaled range analysis for DEM/USD during October 1992. The price is taken as the average of bids and asks over a short sequence of ticks. From Figure (a) to (d), the block consists of 1, 2, 4 and 8 ticks respectively. The dotted curves are formed by the scrambled price sequences and their slopes equal to 0.5. The lower slopes of solid curves in Figure (a) and (b) suggest mean-reverting price behavior. The higher slopes of solid curves in Figure (c) and (d) suggest mean-averting price behavior.

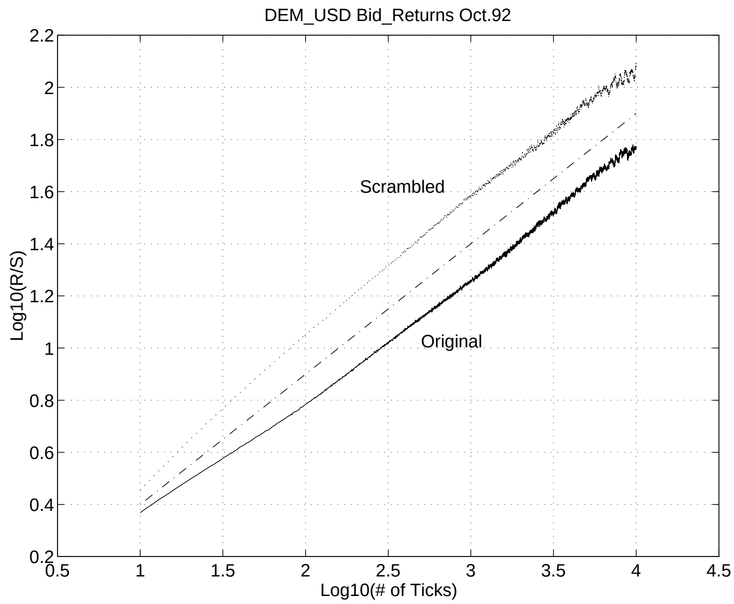


Figure 1: Rescaled range analysis for DEM/USD Bid returns during October 1992. Between Original and Scrambled curves is a straight line with a slope 0.5. The lower slope for the original series for time scales less than 1000 ticks suggests that mean reversion is present on short time scales.

One of our studies is to observe the behavior of the block averages of prices. The block average price is defined as:

$$p_k(t) = \frac{1}{2k} \sum_{i=0}^{i=k-1} (\log(Bid(t-i)) + \log(Ask(t-i))) \tag{7}$$

where k is the length of the sequence of ticks over which the mean price is calculated. The sequences are then downsampled by a factor of k, so that the blocks do not overlap each other.

Rescaled range analysis of the block average prices is presented in Figure 2. Here, the mean prices are calculated for blocks of 1, 2, 4 and 8 ticks respectively. As explained in the figure, the price behavior changes completely for blocks of 1 tick versus blocks of 8 ticks; the behavior in the tick-by-tick DEM/USD series shifts from mean-reverting to mean-averting.

In summary, the mean-reverting price behavior appears to be due to the high frequency oscillations. When short-run oscillations (possibly caused by inventory effects) are smoothed, the price movements shift from mean-reverting to mean-averting

Assuming that a scaling law exists for $[R/S](N)$, we can write

$$[R/S](N) \approx (aN)^H \quad , \tag{6}$$

where a is a constant and H is referred to as the Hurst exponent. By estimating H, we can characterize the behavior of time series as follows:

$$\begin{array}{ll} H = 0.5 & \text{random walk} \\ H \in (0, 0.5) & \text{mean-reverting} \\ H \in (0.5, 1) & \text{mean-averting} \quad . \end{array}$$

For a more detailed, but readable, discussion of R/S analysis and Hurst exponents, see Feder (1988).

2.2. R/S Analysis for High Frequency FX Data

High frequency interbank FX data consists of a sequence of Bid/Ask prices quoted by various firms that function as market makers. While Bid/Ask price quotes from many market makers are displayed simultaneously by wire services such as Reuters and Telerate, a single price series can be constructed from the sequence of newly updated quotes.

We are analyzing a full year of such tick-by-tick Interbank FX price quotes for three exchange rates: the Deutschmark / US Dollar rate (DEM/USD), the Japanese Yen / US Dollar rate (JPY/USD), and the Deutschmark / Yen (DEM/JPY) cross-rate. The data were obtained from Olsen & Associates of Zürich. The data sample includes every tick from October 1992 through September 1993. For the DEM/USD, the year has 1,466,946 ticks.

To study the behavior of returns on a spectrum of time scales, we perform rescaled range analysis and compute Hurst exponents. Figure 1 shows the rescaled range analysis for October 1992 DEM/USD Bid returns. Both the original data and scrambled data were analyzed. The upward shift in the curve for the scrambled data is evidence for mean reversion of the original series on all time scales measured.

Further results are presented in (Moody & Wu (1994)). To summarize them, the behavior of the Hurst exponents in the returns of DEM/USD exchange rates is qualitatively different from the Hurst exponents of gaussian series and scrambled series of the returns. The behaviors of $[R/S](N)$ and H are more similar to those of an AR(1) process with negative coefficient.

To understand the nature and meaning of the apparent mean reverting behavior in the high frequency FX data, we have performed a series of investigations as described in (Moody & Wu (1995*b*)). The question is whether the observed mean reversion over a range of intermediate times scales is due to short-term price oscillations on time scales of a few ticks or is evidence of intrinsic dependencies in the price movements on those intermediate time scales.

scales.

The "Rescaled Range" (R/S) analysis and Hurst Exponents were first developed by Hurst (1951) and refined and popularized by Mandelbrot et.al. in the late 1960's and early 1970's. These became popular in finance due to the clear exposition of the methods in Feder (1988) and the empirical work of Peters (1989). A related approach based on the drift exponent was independently pioneered by Müller *et al.* (1990), and scaling laws for directional change frequency have been suggested by Guillaume (1994). In this paper, we restrict our attention to R/S analysis and Hurst exponents.

2.1. R/S Analysis and Hurst Exponents

The R/S statistic is the range of partial sums of deviations of a time series from its mean rate of change, rescaled by its standard deviation. Denoting a series of returns (one period changes) by r_t, the average m and (biased) standard deviation S of the returns from $t = t_0 + 1$ to $t = t_0 + N$ are:[a]

$$m(N, t_0) = \sum_{t=t_0+1}^{t_0+N} r_t / N \quad , \tag{1}$$

$$S(N, t_0) = \left\{ \frac{1}{N} \sum_{t=t_0+1}^{t_0+N} \left[r_t - m(N, t_0) \right]^2 \right\}^{1/2} \quad . \tag{2}$$

The partial sum of deviations of r_t from its mean and the range of partial sums are then defined as:

$$X(N, t_0, \tau) \equiv \sum_{t=t_0+1}^{t_0+\tau} (r_t - m(N, t_0)) \ \text{ for } \ 1 \leq \tau \leq N \quad , \tag{3}$$

$$R(N, t_0) \equiv \max_{\tau} X(N, t_0, \tau) - \min_{\tau} X(N, t_0, \tau) \quad , \tag{4}$$

The R/S statistic for time scale N is simply the ratio between the average values of $R(N, t_0)$ and $S(N, t_0)$:

$$[R/S](N) \equiv \frac{\sum_{t_0} R(N, t_0)}{\sum_{t_0} S(N, t_0)} \quad . \tag{5}$$

[a]The quantity $S(N, t_0)$ conventionally used in R/S analysis is an estimate of the standard deviation that is biased downward by a factor $\sqrt{(N-1)/N}$. The unbiased estimate of the true standard deviation $\sigma(N, t_0)$ is:

$$\widehat{\sigma}(N, t_0) = \left\{ \frac{1}{N-1} \sum_{t=t_0+1}^{t_0+N} \left[r_t - m(N, t_0) \right]^2 \right\}^{1/2} \quad .$$

In section 5, we present improved results using the unbiased estimate $\widehat{\sigma}(N, t_0)$.

associated with rescaled range analysis and the estimation of Hurst exponents: (1) estimation errors exist when the time scale is very small or very large relative to the number of observations in the time series, (Mandelbrot & Wallis 1969, Wallis & Matalas 1970, Feder 1988, Ambrose, Ancel & Griffiths 1993, Moody & Wu 1995*a*, Müller, Dacorogna & Pictet 1995), and (2) the rescaled range is sensitive to short-term dependence (McLeod & Hipel 1978, Hipel & McLeod 1978, Lo 1991). The second shortcoming will sometimes lead to completely incorrect results.

Lo (1991) analyzed the mean bias in the range statistic R due to short-term dependencies in the time series, and proposed a modified rescaling factor $\tilde{S}$ that is intended to remove or reduce these effects. We have found, however, that Lo's statistic is itself biased and causes some new problems on short time scales while attempting to correct the mean bias of the range R, including distortion of the Hurst exponents. While Lo's approach focuses on the actual value of the $[R/\tilde{S}](N)$ statistic for a given time scale of interest N, Hurst and Mandelbrot test for long term dependency by comparing the slope of $[R/S](N)$ curve to 0.5. Our empirical results show, however, that Hurst exponents, standard rescaled range analysis, and Lo's modified rescaled range can yield incompatible results (with the conventional interpretations of these statistics) due to the biases contained in the R, S, and $\tilde{S}$ statistics.

We propose a new, *unbiased* rescaling factor S^* that is able to correct for the mean biases in R at all time scales without inducing new distortions of the rescaled range and Hurst exponents at short time scales.

The outline of this paper is as follows. In Section 2, we will briefly introduce the rescaled range analysis and the Hurst exponent. The analysis and estimation procedures are then demonstrated on tick by tick interbank foreign exchange data. Through empirical comparisons, we show how seriously short-term dependencies in a time series can affect the rescaled range analysis. In Section 3, we explain why there is a mean bias in the range estimation and introduce Lo's modified approach. Some simulation results with the modified algorithm are shown and compared to results using the original algorithm. In Section 4, we evaluate Lo's modified rescaled range analysis, list and analyze the problems associating with it, and show how it distorts the Hurst exponent. In Section 5, we present our new, unbiased rescaling factor S^* along with empirical results that demonstrate the improvements that it yields relative to the standard R/S and Lo's $R/\tilde{S}$ statistics. In Section 6, we conclude our paper with a discussion.

2. R/S Analysis for High Frequency FX Data

Among the various approaches for quantifying correlations and deviations from gaussian behavior for stochastic processes, several approaches have been suggested that are based on scaling laws. Unlike traditional correlation analysis, these scaling law methods are intended to quantify structure that persists on a spectrum of time

Appears in "Neural Networks in the Capital Markets",
Proceedings of the Third International Conference (London, October 1995),
A. Refenes, Y. Abu-Mostafa, J. Moody, and A. Weigend, eds., World Scientific, London, 1996.

IMPROVED ESTIMATES FOR THE RESCALED RANGE AND HURST EXPONENTS

John Moody and Lizhong Wu
Computer Science Dept., Oregon Graduate Institute,
Portland, OR 97291-1000
E-mail: moody@cse.ogi.edu, lwu@cse.ogi.edu

ABSTRACT

Rescaled Range R/S analysis and Hurst Exponents are widely used as measures of long-term memory structures in stochastic processes. Our empirical studies show, however, that these statistics can incorrectly indicate departures from random walk behavior on short and intermediate time scales when very short-term correlations are present. A modification of rescaled range estimation ($R/\tilde{S}$ analysis) intended to correct bias due to short-term dependencies was proposed by Lo (1991). We show, however, that Lo's $R/\tilde{S}$ statistic is itself biased and introduces other problems, including distortion of the Hurst exponents. We propose a new statistic R/S^* that corrects for mean bias in the range R, but does not suffer from the short term biases of R/S or Lo's $R/\tilde{S}$. We support our conclusions with experiments on simulated random walk and AR(1) processes and experiments using high frequency interbank DEM / USD exchange rate quotes. We conclude that the DEM / USD series is mildly trending on time scales of 10 to 100 ticks, and that the mean reversion suggested on these time scales by R/S or $R/\tilde{S}$ analysis is spurious.

1. Introduction and Overview

There are three widely used methods for long-term dependence analysis: autocorrelation analysis, fractional difference models (Granger & Joyeux 1980, Hosking 1981), and scaling law analysis, including rescaled range (R/S) analysis (Hurst 1951), Hurst exponents, (Hurst 1951, Mandelbrot & Van Ness 1968), and drift exponents (Müller, Dacorogna, Olsen, Pictet, Schwarz & Morgenegg 1990). This paper studies R/S analysis and Hurst exponents, which have become recently popular in the finance community largely due to the empirical work of Peters (1989). Compared to autocorrelation analysis, the advantages of R/S analysis include: (1) detection of long-range dependence in highly non-gaussian time series with large skewness and kurtosis, (2) almost sure convergence for stochastic processes with infinite variance, and (3) detection of nonperiodic cycles. However, there are also two deficiencies