Financial Economics (I)
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Lecture 7: Rescale Range Analysis and the Hurst Exponent

Hurst exponent is one of the most frequently used statistics to describe the the weakly stationary stochastic process which has long memory. In this lecture, the classical rescaled range statistic of Hurst is computed for seven Asia-Pacific countries' stock indices. It is confirmed that all index return have long memory.

Introduction

- A weakly stationary process has *short memory* when its *autocorrelation function* (ACF) $\rho(.)$ is geometrically bounded, i.e., there exists a constant C such that
$$\mid \rho(h) \mid \leq C\gamma^{|h|} \tag{1}$$
where $0 < \gamma < 1$.

- The stationary stochastic processes frequently referred in financial time series, such as *ARCH* (Engle, 1982), *GARCH* (Bollerslev, 1986), *IGARCH* (Engle and Bollerslev, 1986), and *EGARCH* (Nelson, 1991) all have short memory for volatilities.

- A weakly stationary process have *long memory* if its ACF has a hyperbolic decay,
$$\rho(h) \sim Ch^{2d-1} \quad as \quad h \to \infty, \tag{2}$$
where $C \neq 0$ and $d < 0.5$. The stationary stochastic processes such as *LMSV* (Bredit, Crato and de Lima, 1994), *FIGARCH* (Baillie, Bollerslev and Mikkelsen, 1996), *FIEAGRCH* (Nelson, 1991; Bollerslev and Mikkelsen, 1996) are long-memory models for volatilitis.

- One of the most interesting findings in financial econometrics is that many financial time series has *long memory*. For example, Bollerslev and Mikkelsen (1996) found slowly decaying autocorrelations for the absolute returns $| r_t |$ of the Standard & Poor's 500 index. Also, Baillie, Bollerslev and Mikkelsen (1996) found persistence in the volatility of nominal exchange rates.

- Persistence in autocorrelations was first found in hydrological data by Hurst (1951). This phenomenon in river flow time series was named *Hurst effect*. It is cited in Mandelbrot and Wallis (1969) as the *Joseph effect*.

- In the biblical story, Joseph interpreted Pharaoh's dream to mean seven years of plenty followed by seven years of famine.

- Mandelbrot and Wallis (1969) suggested to model such time series with *fractional Gaussian noise process*. It can be shown that the Hurst exponent (breifly H), as defined in the next section, is exactly the fundamental parameter of the *fractional Gaussian noise process*. An H of 0.5 implies a non-detrministic process, i.e., one in which the past history does not influence the future course of the series. An H of less than 0.5 implies *anti-persistent* behaviour. An H of greater than 0.5 implies a *persistent* behavior.

Rescaled Range Analysis: The Algorithm Peters (1994) provides the basic procedure for calculating *Hurst exponents*. The following algroithm is adapted from Hampton (1996a).

1. For a given financial time series of interest, say the Taiwan Stock Exchange Index P_t, select T samples of the series, where T is greater than the longest cycle of interest.

2. From the T samples of data, construct $T-1$ variables of the logarithmic returns of sequential data points as follows

$$r_t = \ln^{\frac{P_{t+1}}{P_t}} \tag{3}$$

3. Begining with the smallest value of n, partition the data into T_n sequential non-overlapping blocks, where $T_n = \frac{T-1}{n}$. If T_n is not interger, then it is redefined as $[\frac{T-1}{n}]$, where [ ] is the Gauss symbol.

4. Compute the local mean rate of return for each block of data.

$$\bar{r}_{[i:T_n]} = \frac{\sum_{t=(i-1)n+1}^{in} r_t}{n}, i = 1, 2, ...T_n \tag{4}$$

5. Compute the local standard deviation (*local volatility*), $S_{[i:T_n]}$

$$S_{[i:T_n]} = \sqrt{\frac{\sum_{t=(i-1)n+1}^{in} (r_t - \bar{r}_{[i:T_n]})^2}{n}} \tag{5}$$

6. Compute the accumulated differences between each r_t and the corresponding $\bar{r}_{[i:T_n]}$ for each block.

$$D_{j,i} = \sum_{t=(i-1)n+1}^{(i-1)n+j} (r_t - \bar{r}_{[i:T_n]}), i = 1, 2, ...T_n, j = 1, 2, ..., n \tag{6}$$

7. Substract the minimum from the maximum of the accumulated difference for each block, providing the local range, $R_{[i:T_n]}$

$$R_{[i:T_n]} = max(D_{j,i}) - min(D_{j,i}), i = 1, 2, ...T_n, j = 1, 2, ..., n \tag{7}$$

8. Divide each local $R_{[i:T_n]}$ by each corresponding local $S_{[i:T_n]}$, producing

$$(\frac{R}{S})_{[i:T_n]} = \frac{R_{[i:T_n]}}{S_{[i:T_n]}} \tag{8}$$

   This value is referred to as the *rescaled range.*

9. Compute the average of $\frac{R}{S}_{[i:T_n]}$,

$$(\frac{R}{S})_{T_n} = \frac{\sum_{i=1}^{T_n} (\frac{R}{S})_{[i:T_n]}}{T_n} \tag{9}$$

10. Repeat steps 3 through 9 using the next large value of n until $n = \frac{N}{2}$.

11. Plot the $\ln^{(\frac{R}{S})_{T_n}}$ versus the $\ln^n$ for each n.

12. Generate a linear regression through the plot.

$$\ln^{(\frac{R}{S})_{T_n}} = K + H \ln^n \tag{10}$$

13. Compute the slope of the regression line, provdiding the estimated Hurst exponent for a range of n, denoted by H_n.

14. Hurst exponents for individual values of n can be estimated by

$$H_n = \frac{\ln^{(\frac{R}{S})_{T_n}}}{\ln^n} \tag{11}$$

15. The program to implement the rescaled range analysis can be downloded from here.

Data Description

- The data employed in this paper is summarzied in Table 1. The R/S analysis is applied to seven Asia-Pacific stock markets. Except the U.S., all data used are daily indices. For U.S., the monthly data is used.

- The reason of using data with different frequencies is mainly motivated by a series studies which indicate that *information contained in the data tends to disappear when high-frequency data is replaced by the low-frequency data.*

- If this is ture, then the Hurst exponet of the high-frequency data is expected to be higher than that of the low one.[1]

[1]More precisely, consider the Hurst exponent H_{T_n} as the function of the frequency (ω) of the observation. Then the observation above is simply

$$\lim_{\omega \to 0} H_{T_n}(\omega) = \frac{1}{2}$$

.

Table 1: The Stock Index of Seven Asia-Pacific Countries

Country	Index	Period	T
Taiwan	TAIEX (daily)	1/1/71 - 7/1/94	6820
Taiwan	TAIEX (monthly)	1/1/71 - 7/1/94	227
U.S.	S&P 500	1/71 - 12/93	275
Australia	All Ordinary Share	1/2/80 - 3/36/97	4320
Malaysia	KLSE	5/1/92 - 3/26/97	1211
Indonesia	Aktienkursindex	11/1/90- 3/26/97	1620
Thailand	Bangkok SET	6/16/76 -3/26/97	5326
Philippine	Manila Comp. Share	1/2/86 - 3/26/96	2873

Data Source: AREMOS DATABASE

Empirical Results (I)

- While the Hurst exponent can be obtained from Equation (10), Peters (1996) also cautions researchers against running the OLS regression for the entire range of n.
- Some systems may have short natural memory cycles, as a consequence, their $\frac{R}{S}$ estimates would display a persistence only for the short subperiods. Thus, one must first observe the $\ln^{(\frac{R}{S})_{Tn}}$ by $\ln^{n}$ plots and then obtain an H estimate only for the *lineaer portion of the plot.*
- In this lecture, we adopt the above-mentionded technique.
- Another caution made by Ambrose, Ancel and Griffiths (1993) is that regression analysis, which weights equally the few samples at relatively long time scales and those of much shorter time scales, biases the results in favor of finding short-term behaiour in the time series.
- The OLS regression (10) is run with respect to different ns, namely, $n = 150, 300, 450, ...$ up to approximately one half of the total number of observations.
- The Matlab program to compute the Hurst exponent is availabe upon request.
- The Hurst exponent estimated under different ns is then plotted with the associated n in Figures 1-5.

Table 2: The Hurst Exponent of Five Asia-Pacific Stock Markets

Country	Husrt Exponent	Cycle (Trading Days)
Australia	0.7000 (0.6833)	1350
Malaysia	0.7468	300
Indonesia	0.7940	750
Thailand	0.6494	2250
Philippine	0.6725	1200

Inside the bracket are the corresponding Hurst exponents estimated by Pandey, Kohers and Kohers (1995). The sample used in this that research consists of the *weekly* national stock indices retrived from *Morgan Stanley Capital International Perspective* of Geneva, Switzerland from Feb. 23, 1978 through July 1, 1994.

- By choosing the highest point of these plots, the Hurst Exponet for Australia, Phillipine, Indonesia, Malaysia and Thailand is given in Table 2.

Some Comments:

- Of course, a prudent decision cannot be made until we can show that these values are statistically significantly different from 0.5.
- However, there is no finite-sample distribution theory of the classical R/S statistic. As Ravenna (1996) put, "Being so difficult to estimate the total bias, it is not possible to conceive a rigorous test for Hurst coefficients' significance." (ibid, p. 15).
- Peters (1994) provides some empircial distributions about the Hurst exponent based on Monte Carlo experiments. See Peters (1994), pp.71-74.
- In practice, "heursitics are sometimes developed where estimates of Hurst exponets which fall within an empirically-derived range, say [0.45, 055], are interpreted as representative of a random walk process (Hampton, 1996, p25)".

Table 3: The Hurst Exponent of Taiwan Stock Market

Frequency	Husrt Exponent	Cycle
Daily	0.6337	4.5 yrs
Monthly (Monthly Closing Price)	0.6371	5 yrs
Monthly (Simple Avergae)	0.6678	4 yrs

Empirical Results (II)

- In the second experiment, we apply the R/S analysis to the Taiwan stock index with different frequencies, namely, the daily data and monthly data.

- Basically, we would like to see the robustness of the Hurst exponents estimated under different frequencies of the data.

- In this experiment, we consider two types of monthly data. The first type is simply the *closing* price of the month, and the second is the *simple average* of the daily index. The Hurst exponents estimated are also given in Table 3 and are plotted in Figures 6-8.

- From Table 3, we can see that the resultant Hurst exponents estimated under different frequencies and different transformation of data are not the same.

- Moreover, an inspection of Figure 7 and Figure 8 reveals that the Hurst exponent cannot be uniquely determined.

- For example, in Figure 7, the estimated Hurst exponent can be 0.6434, 0.6405 or 0.6371. While these three estimates are not dramtically different, the resultant estimation of the length of cycles can range from 2 years to 5 years. Among them, the one with 5-year cycle is closed to the result from daily data, i.e., 4.5 years. If we choose this one, the Hurst exponent is 0.6371. Following the same criterion, the Hurst exponent of the simple average of one-month daily index return is 0.6678 with a 4-year cycle.

Empirical Results (III)

- In the last experiment, the R/S analysis is applied to the S&P 500.
- S&P 500 has been extensively studied by many researchers. For example, using monthly returns from Jan. 1950 to July 1988, Peters (1992) finds that the Hurst exponent for S&P 500 is 0.78. However, it is estimated to be 0.5849 in Pandey, Kohers and Kohers (1992) who use the weekly data.
- In this paper, we use monthly returns, while with a different sample period, and come up with the estimate 0.687 (See Figire 9).
- Hampton (1996b) summarizes the results of various methods for interpreting Hurst exponents. He finds that the results are not consistent in their characterization of the S&P 500. He concludes that the work by Peters (1994) is not confirmed regarding daily returns for the S&P 500 over the value of n studied in this work.

Concluding Remarks:

- The R/S analysis of seven Asia Pacific stock markets show that stock return display long memory. However, this analysis also indicates the robustness of the estimated Hurst exponent.
- First, we find that the estimated Hurst exponent may be sensistive to the frequencies of the data employed, daily, weekly, monthly,...etc.
- Second, the estimated Hurst exponent is not invariant to different sample period. Nevertheless, if we do not restrict our attention to the Hurst coefficient only, then the qualitative result of long memory is valid for all cases.
- Long memory may have important implications for the construction of trading systems. Hampton (1996c) illustarted the use of the Hurst exponent for trading. In his particular example, he use the 10-day average of the estimated Hurst exponent as a technical index. If this index is higher than 0.66, then it is time to buy. If this index is lower

than 0.43, then it is time to sell. One of future direction for research is how to make the use of the property of long memory to design trading strategies.

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